

Mortality



The idea of collecting statistics about causes of death dates back at least to the late 16th century. During an outbreak of the plague at this time the English government started publishing weekly death statistics in 'Bills of Mortality'. In 1662 John Graunt used the data from thirty years of these Bills to make observations and predictions. His book 'Observations on the London Bills of Mortality' included calculations involving the high rates of death in infancy and in London in comparison to the whole of England. John Graunt also made realistic estimates of the size of the population of London, showing that the population was increasing. Here is an extract from his book:

"We have (though perhaps too much at Random) determined the number of the inhabitants of *London* to be about 384000: the which being granted, we assert, that 199112 are *Males*, and 184886 *Females*.

Where as we have found, that of 100 quick Conceptions about 36 of them die before they be six years old, and that perhaps but one surviveth 76, we, having seven *Decads* between six and 76, we sought six mean proportional numbers between 64, the remainder, living at six years, and the one, which survives 76, and finde, that the numbers following are practically near enough to the truth; for men do not die in exact Proportions, nor in Fractions: from whence arises this Table following.

<i>Viz.</i> of 100 there dies		The fourth	6
within the first six years	36	The next	4
The next ten years, or		The next	3
<i>Decad</i>	24	The next	2
The second <i>Decad</i>	15	The next	1
The third <i>Decad</i>	09		

From whence it follows, that of the said 100 conceived there remains alive at six years end 64.

At Sixteen years end	40	At Fifty six	6
At Twenty six	25	At Sixty six	3
At Tirty six	16	At Seventy six	1
At Fourty six	10	At Eight	0

It follows also, that of all, which have been conceived, there are now alive 40 *per Cent.* above sixteen years old, 25 above twenty six years old, & *sic deinceps*, as in the above Table: there are therefore of Aged between 16, and 56, the number of 40, less by six, *viz.* 34; of between 26, and 66, the number of 25 less by three, *viz.* 22: *sic deniceps*. Wherefore, supposing there be 199 112 *Males*, and the number between 16, and 56, being 34. It follows, there are 34 *per Cent.* of all those *Males* fighting Men in *London*, that is 67 694, *viz.* near 70 000...."

A few years later Sir Edmund Halley (of comet fame), used Graunt's ideas to create the first actuarial tables for the developing life insurance industry.

The practice of using mortality rates in life insurance continues to this day, but the theory from which mortality rates are predicted has become more sophisticated since John Graunt's time.





Benjamin Gompertz
Source: www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Gompertz.html

In 1825 a British actuary, Benjamin Gompertz, discovered a pattern in human mortality. He found that the probability of dying was high at birth but then declined until sexual maturity. After this it increased at an exponential rate. He estimated that between the ages of 20 and 80 years, the risk of dying doubles with every additional 8 years that you live.

Gompertz's law of mortality can be written as $\mu_x = ae^{bx}$ where the 'force of mortality', μ_x , is given in terms of age, x years.

μ_x is the instantaneous death rate i.e. the number of deaths per head of population per unit time. It is also known as the 'hazard rate'.

When the constants a and b are greater than zero, ae^{bx} is an increasing function of x .

Gompertz's law can also be expressed in the linear form $\ln \mu_x = \ln a + bx$

Assuming that the risk of dying doubles every eight years then $\mu_{x+8} = 2\mu_x$, giving $ae^{b(x+8)} = 2ae^{bx}$. This equation can be solved to give an approximate value for b of 0.0866.

In 1860 William Makeham suggested that Gompertz's law could be improved by adding a constant term to give $\mu_x = ae^{bx} + c$. This is usually referred to as the Gompertz-Makeham mortality law. It has been found to fit adult populations well, with variations in the parameters a , b and c allowing for differences between populations.

For example, values that have been used by Danish insurance companies are: $a = 0.000\ 0759$, $b = 0.0875$ and $c = 0.0005$.

Explanations of why the Gompertz-Makeham mortality law works usually relate the constant c to the risk of death from all causes that do not depend on age, whilst the term ae^{bx} is related to the risk of death because of the deterioration of the body due to ageing processes.

In more recent years other mathematical models have been suggested, in particular the logistic model,

$\mu_x = \frac{ae^{bx}}{1 + \alpha e^{bx}} + c$, suggested by Perks in 1932. This includes the Gompertz-Makeham law as a special case when $\alpha = 0$.

It is interesting to note that the theory of mortality has also been found useful in other contexts. For example, the Gompertz model can be used to model the failure of technical systems due to wear and tear.



Mortality



Comprehension Questions

- 1 a According to John Graunt what was the probability of a baby
- i living until at least the age of 56 years ii dying before the age of 46 years (2 marks)
- b The second column of this table shows the number of people from the original group of 100 babies that John Graunt expects to die in the corresponding age range. Complete the third column to show the percentage of those alive at the beginning of the age range that he expects to die before they reach the end of the age range. (3 marks)
- | Age (years) | No. dying | % dying |
|-------------|-----------|---------|
| 0 – | 36 | 36% |
| 6 – | 24 | 37.5% |
| 16 – | 15 | |
| 26 – | 9 | |
| 36 – | 6 | |
| 46 – | 4 | |
| 56 – | 3 | |
| 66 – | 2 | |
| 76 – | 1 | |
- c Starting with the figures in John Graunt's second table, show calculations to explain how he came to the conclusion that there are approximately '70 000 fighting men' in London. (2 marks)
- 2 Gompertz's law of mortality can be written as $\mu_x = ae^{bx}$. Makeham suggested that Gompertz's law could be improved by adding a constant term to give $\mu_x = ae^{bx} + c$.
- a Sketch graphs of $\mu_x = ae^{bx}$ and $\mu_x = ae^{bx} + c$ for $x \geq 0$ assuming that $a > 0$, $b > 0$, and $c > 0$. (2 marks)
- b The article states of the Gompertz function, $\mu_x = ae^{bx}$, that 'When the constants a and b are greater than zero, μ_x is an increasing function of x .'
- i Explain what this means and how it is shown by your graph. (1 mark)
- ii If $a > 0$, what values of b would make $\mu_x = ae^{bx}$ a decreasing function of x ? (1 mark)
- c Use the Gompertz-Makeham function $\mu_x = ae^{bx} + c$ with $a = 0.000\ 0759$, $b = 0.0875$ and $c = 0.0005$ to calculate the value of μ_x when i $x = 20$ and ii $x = 80$ (2 marks)
- 3 a Show how Gompertz's law, $\mu_x = ae^{bx}$, can be written in the form $\ln \mu_x = \ln a + bx$ (2 marks)
- b Sketch a graph of $\ln \mu_x$ against x for $x \geq 0$, assuming that $0 < a < 1$ and $b > 0$. Give the intercept on each axis in terms of a and b . (3 marks)
- 4 The article states that: 'If the risk of dying doubles every eight years then $\mu_{x+8} = 2\mu_x$, giving $ae^{b(x+8)} = 2ae^{bx}$. This equation can be solved to give an approximate value of 0.0866 for b .' Show all the steps in the mathematical argument that gives this value for b . (4 marks)
- 5 Give one example of a cause of death that is likely to contribute to each of the terms in the Gompertz-Makeham function $\mu_x = ae^{bx} + c$ (2 marks)
- (Total 24 marks)



Teacher Notes

Unit Advanced Level, Applying mathematics

Notes

It is intended that the article should be discussed with students before they attempt the questions. They will probably need help with the excerpt from John Graunt's book which is quite difficult to follow. The first entry in the first table shows that of 100 babies ('Conceptions'), he would expect 36 to die before they reached the age of 6 years. The rest of the table shows how many of the survivors he would expect to die in following decades ('decads'). In working this out he assumes that the percentage of survivors who die during each decade is roughly the same (about 37%). However the rounding needed to give whole numbers of people causes variation in this percentage, especially in the older age ranges. The second table shows how many of the original 100 babies, Gompertz would expect to survive to each of the given ages.

Answers

- 1 a i P(living until at least 56 years) = 0.06 ii P(dying before 46 years) = 0.9
 b

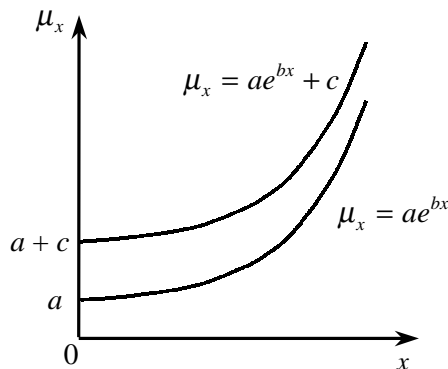
Age (years)	No dying	% dying
0 –	36	36%
6 –	24	37.5%
16 –	15	37.5%
26 –	9	36%
36 –	6	37.5%
46 –	4	40%
56 –	3	50%
66 –	2	66.7%
76 –	1	100%

- c 'Fighting men' are those aged between 16 and 56.

Since 40% of people are above 16 years old and 6% above 56 years old, he estimates that 34% of the population are between these ages.

34% of 199112 males in London = 67 698

- 2 a



- b i As x increases μ_x increases.
 The graph has a positive gradient at all points.

ii $b < 0$

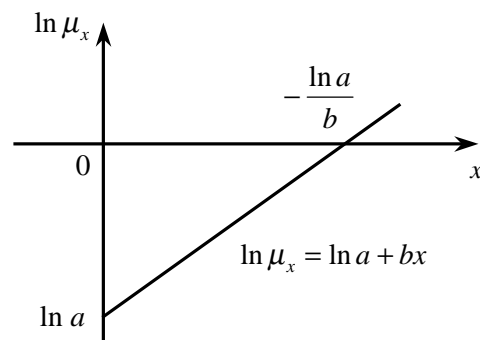
- c i When $x = 20$, $\mu_x = 0.000937$
 ii When $x = 80$, $\mu_x = 0.0837$

- 3 a Taking natural logs gives

$$\ln \mu_x = \ln(ae^{bx}) = \ln a + \ln e^{bx}$$

$$\Rightarrow \ln \mu_x = \ln a + bx$$

- b



$$\begin{array}{llll}
 4 \text{ Assuming } a \neq 0, & ae^{b(x+8)} = 2ae^{bx} & \Rightarrow & e^{b(x+8)} = 2e^{bx} \\
 & & \Rightarrow & e^{(bx+8b)} = 2e^{bx} \\
 & & \Rightarrow & e^{bx} \cdot e^{8b} = 2e^{bx} \\
 \text{Since } e^{bx} \neq 0, & & & e^{8b} = 2 \\
 & & \Rightarrow & 8b = \ln 2 = 0.6931 \\
 & & \Rightarrow & b = 0.0866 \text{ (to 3 sf)}
 \end{array}$$

- 5 Any cause of death that does not depend on age contributes to the constant c (eg accidental death)
 Any cause of death that is related to the deterioration of the body due to ageing processes contributes to the exponential term ae^{bx} (eg heart disease)

Marks

The comprehension has a total of 24 marks.

This reflects the number of marks available in the comprehension paper (Paper 1) for the Applying Mathematics unit. In that paper there another 6 marks available for mathematics presented:

- accurately using correct notation
- logically and clearly.

